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Indirect Spacecraft Trajectory Optimization Using Modified Equinoctial Elements

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Introduction

SEVERAL different types of orbital elements have been used over the years for spacecraft trajectory design. Four frequently used orbital elements are Cartesian elements (CE), Keplerian elements (KE), equinoctial elements (EE), and modified equinoctial elements (MEE), the details of which are given in [1]. CE-based equations are singularity free but are computationally difficult to solve; KE and EE equations can be solved efficiently but have singularities at an inclination or eccentricity of 0 (KE equations) or an eccentricity of 1 (EE equations).

MEE-based equations of motion are singularity free for all trajectories with inclinations of less than 180 deg and exhibit the same computational advantages as equations based on EE or KE. For these reasons, MEE-based equations of motion are good candidates for use in low-thrust escape trajectory design.

To our knowledge, a complete treatment of escape trajectory design making use of MEE-based adjoint equations incorporating the third-body and higher-order Earth gravitational perturbations has not appeared previously in the literature, although previous researchers have looked at more restricted aspects of the total problem [2]. To implement trajectory design incorporating the perturbations, the Jacobian matrix of the Earth-centered inertial (ECI) state vector $[\mathbf{r}^T \quad \mathbf{v}^T]^T$ is derived and its elements are presented in this Note. As an example, a low Earth orbit (LEO) to hyperbolic orbit transfer trajectory with J_2 to J_6 Earth gravity and sun–moon gravitational perturbations is solved using an indirect method.

Equations of Motion for Flight Under Small Perturbing Forces

The defining equation for spacecraft motion under small perturbing forces written in terms of MEE [3] is as follows:

$$\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}t} = B(\mathbf{P} + \mathbf{G}) + \mathbf{b} \tag{1}$$

where $\mathbf{Z} = [p \quad f \quad g \quad h \quad k \quad L]^T$, and the symbols used for the state variables are same as those used in [3]. (A third term must be added into the parenthesis of the first term on the right-hand side of Eq. (1) if other perturbing accelerations such as radiation pressure and aerodynamic drag are considered. In this Note, only gravitational and thrust accelerations are considered.) $\mathbf{P}$ is the perturbing acceleration due to thrust, $\mathbf{b}$ is the time rate of change of $\mathbf{Z}$ when perturbing forces are not present, and $\mathbf{G}$ is the perturbing acceleration due to the higher-order gravitational terms of Earth and additional celestial bodies. Both $\mathbf{P}$ and $\mathbf{G}$ are expressed in the rotating RSW coordinate system as described in [4]. The equations for the matrix B and vector $\mathbf{b}$ written in terms of MEE are given by [3].

Mathematical Statement of the Time-Optimal Problem

The time-optimal problem for spacecraft flight is given as follows:

$$\text{maximize} \quad -t_f \tag{2}$$

$$\text{subject to} \quad \frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}t} = \mathbf{F}(t, \mathbf{Z}, \mathbf{P}) = B(\mathbf{P} + \mathbf{G}) + \mathbf{b} \tag{3}$$

with boundary conditions

$$Z_i(t_0) = Z_i^{(0)} \quad \text{with} \quad 1 \le i \le 6 \tag{4}$$

$$Z_j(t_f) = Z_j^{(f)} \quad \text{with} \quad 1 \le j \le 6 \tag{5}$$

where t_f is the time of flight, and $Z_i^{(0)}$ and $Z_j^{(f)}$ are the given ith and jth components of the state vector at the initial and final times, respectively. It is important to note that the number of initial states set by Eq. (4) does not need to equal the number of final states set by Eq. (5).

Optimal Control of Ordinary Differential Equation

The Hamiltonian of the dynamical system given by Eq. (3) is

$$H = \boldsymbol{\lambda}^T \mathbf{F}(t, \mathbf{Z}, \mathbf{P}) \tag{6}$$

Pontryagin's maximum principle [5] implies that the optimal control law and trajectory solve the system of differential-algebraic equations (7–9):

$$H(\mathbf{P}_{\text{opt}}) = \max_{\mathbf{P}} H(\mathbf{P}) \tag{7}$$

$$\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}t} = \mathbf{F}(t, \mathbf{Z}, \mathbf{P}_{\text{opt}}) = B(\mathbf{P}_{\text{opt}} + \mathbf{G}) + \mathbf{b} \tag{8}$$

$$\frac{\mathrm{d}\boldsymbol{\lambda}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{Z}} \tag{9}$$

The control vector $\mathbf{P}$ can be written as $P \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector pointing in the direction of the thrust vector, and P is the magnitude of $\mathbf{P}$. In this Note, P is assumed to be fixed, so that the optimal control problem becomes a problem of finding a time history of the pointing vector $\mathbf{u}$.

Aside from Eqs. (4) and (5), the system of differential-algebraic equations (7–9) has the following additional boundary conditions, which are derived from the first-order optimality conditions [5] for time-optimal problems:

$$\lambda_m(t_0) = 0, \qquad \text{if the } m\text{th component of } \mathbf{Z}(t_0) \text{ is not given} \tag{10}$$

$$\lambda_n(t_f) = 0, \qquad \text{if the } n\text{th component of } \mathbf{Z}(t_f) \text{ is not given} \tag{11}$$

$$H(t_f) = 1 \tag{12}$$

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Table 1 Initial, end, and achieved values

Parameters	Initial	Target	Achieved at t_f
p, km	7000	88200.000000000015	88199.999375565108
f	0	1.1	1.0999999920423695
g	0	0	$1.2138900187114166 \times 10^{-8}$
h	$\tan(\frac{1}{2}\frac{\pi}{180} \times 28.5)$	$\tan(\frac{1}{2}\frac{\pi}{180} \times 1)$	0.0087268683298500661
k	0	0	$2.1138332143705307 \times 10^{-10}$
L, rad	Free	Free	20.04590043994158
λ_L	0	0	$2.6626795099105038 \times 10^{-4}$

Solving the boundary value problem (BVP) defined by Eqs. (4), (5), and (7–12) involves finding an initial adjoint vector and a t_f such that this system of equations is satisfied. When the initial condition for a state variable is not specified, the initial value of that state variable will be subjected to optimization also. For the rest of this Note, the collection of the initial adjoints, state variables, and the time of flight t_f subjected to optimization will be referred to as the solution of the BVP.

It can be shown geometrically that the $\mathbf{u}$ that leads to Eq. (7) can be expressed in the following way [5]:

$$\mathbf{u}_{\text{opt}} = \frac{B^T \boldsymbol{\lambda}}{\|B^T \boldsymbol{\lambda}\|} \tag{13}$$

Substitution of Eq. (7) into Eq. (8) renders the system of differential-algebraic equations (7–9) into a system of nonlinear ordinary differential equations consisting only of Eqs. (8) and (9).

For the cases in which H is not an explicit function of time, the optimal $\boldsymbol{\lambda}$ and $\mathbf{Z}$ will produce an H that is constant throughout the duration of the flight [5], a fact that is useful for determining the optimality of the control and state vectors.

The gradient of the Hamiltonian is given as follows:

$$\begin{aligned}\left[\frac{\partial H}{\partial \mathbf{Z}}\right]_k &= \left(\left\{\frac{\partial}{\partial \mathbf{Z}}[B(\mathbf{P}_{\text{opt}} + \mathbf{G})] + \frac{\partial \mathbf{b}}{\partial \mathbf{Z}}\right\}^T \boldsymbol{\lambda}\right)_k \\ &= \sum_{i=1}^{6}\sum_{j=1}^{3}\left\{\frac{\partial B_{ij}}{\partial Z_k}(P_j + G_j) + B_{ij}\frac{\partial G_j}{\partial Z_k} + \frac{\partial b_i}{\partial Z_k}\right\}\lambda_i\end{aligned} \tag{14}$$

where

$$\left[\frac{\partial \mathbf{G}}{\partial \mathbf{Z}}\right]_{ik} = \left[\frac{\partial}{\partial \mathbf{Z}}(Q^T \boldsymbol{\Delta}_g)\right]_{ik} = \sum_{j=1}^{3}\left[\frac{\partial Q_{ij}^T}{\partial Z_k}(\Delta_g)_j + Q_{ij}^T \frac{\partial (\Delta_g)_j}{\partial Z_k}\right] \tag{15}$$

$$\left[\frac{\partial \boldsymbol{\Delta}_g}{\partial \mathbf{Z}}\right]_{ik} = \sum_{j=1}^{3}\left[\frac{\partial (\Delta_g)_i}{\partial r_j}\frac{\partial r_j}{\partial Z_k}\right] \tag{16}$$

$$\left[\frac{\partial Q^T}{\partial \mathbf{Z}}\right]_{ijk} = \sum_{p=1}^{3}\left(\frac{\partial Q_{ij}^T}{\partial r_p}\frac{\partial r_p}{\partial Z_k} + \frac{\partial Q_{ij}^T}{\partial v_p}\frac{\partial v_p}{\partial Z_k}\right) \tag{17}$$

where $\boldsymbol{\Delta}_g$ is the sum of the non-Keplerian Earth gravitational perturbations and third-body effects from the sun and the moon, and Q^T is the matrix that transforms a vector from the ECI frame to the RSW frame and can be found in [3].

Deriving $\frac{\partial \mathbf{r}}{\partial \mathbf{Z}}$, and $\frac{\partial \mathbf{v}}{\partial \mathbf{Z}}$ for the MEE-based system is one of the main objectives of this research, and elements of these gradients are presented in the Appendix. $\frac{\partial B}{\partial \mathbf{Z}}$ and $\frac{\partial \mathbf{b}}{\partial \mathbf{Z}}$ are derived in [2], and $\frac{\partial \boldsymbol{\Delta}_g}{\partial \mathbf{r}}$, $\frac{\partial Q}{\partial \mathbf{r}}$, and $\frac{\partial Q}{\partial \mathbf{v}}$ are derived in [6].

Numerical Example

In this section, the solution to an example problem is presented. The problem involves the design of an LEO to hyperbolic orbit transfer trajectory and demonstrates the utility of MEE-based systems in solving problems involving escape trajectories.

The trajectory optimization problem concerned in this Note is solved using an indirect method. In comparison with discretization-based methods such as a direct method [7] and a hybrid method [2], the indirect method yields a solution that is closer to optimum. A major drawback of using an indirect method is the difficulty of obtaining a solution when the initial guesses for adjoints and states (for cases of free initial states) are not close to the actual optimal solution. Indirect methods are therefore useful for cases in which the mission involved is similar to a previous one, or when homotopy approaches are used, as is the case in the example discussed here.

SNOPT [8], a software package written in FORTRAN 77, is the nonlinear package used in this work. The specific version of SNOPT used in this work is designed to interface with MATLAB®. The software solves nonlinear programming problems encountered in this work using a gradient-based search method [9]. All calculations presented in this Note used the default feasibility tolerance of 10^{-6}.

The integrator used for all calculations is MATLAB's ODE45 variable step Runge–Kutta solver. The initial time step is set to 10^{-2} s and the relative and absolute tolerances are both set to 10^{-10}.

There is an important difference between the way in which the trajectory optimization was carried out in this work and the method of solution presented in [9]. In [9], the solution is obtained by minimizing a weighted sum in the following form:

$$\begin{aligned}s = {}& w_1[p(t_f) - p_{\text{target}}]^2 + w_2[f(t_f) - f_{\text{target}}]^2 + w_3[g(t_f) \\ &- g_{\text{target}}]^2 + w_4[h(t_f) - g_{\text{target}}]^2 + w_5[k(t_f) - k_{\text{target}}]^2 \\ &+ w_6[\lambda_L(t_f)]^2 + w_7[H(t_f) - 1]^2\end{aligned} \tag{18}$$

In this Note, a solution is obtained by configuring SNOPT to act only as a constraint solver [8].

The values presented in all of the tables in this section are displayed to twelfth digits of precision after the decimal. The epoch used in the example problem is 1 January 2008, 00:00:00. The P used in the problem is 9.8×10^{-5} km/s^2, which is the value used in [9].

The perturbing accelerations used in the example presented here are the Earth zonal gravitational perturbations ranging from the second to the sixth (J_2 to J_6) orders and the third-body effects due to the sun–moon gravity. The ephemerides of the sun and the moon are calculated using MATLAB codes.[‡] Table 1 lists the initial and target values for the transfer problem, as well as the actual achieved values. To ensure fast convergence for the first homotopy step, the initial and final states used in this example, with the exception of $p(t_f)$, are set to be identical to those used in [9]; moreover, the perigee $r_p(t_f)$ used in this example is also the same as that used in [9].

The solution to the example problem is obtained iteratively. In the first iteration, the solution to the example problem in [9] is set as the initial guess of the optimal solution, but the actual problem solved in

[‡]Data available online at http://wise-obs.tau.ac.il/~eran/MATLAB/Ephem.html [retrieved 19 December 2009].

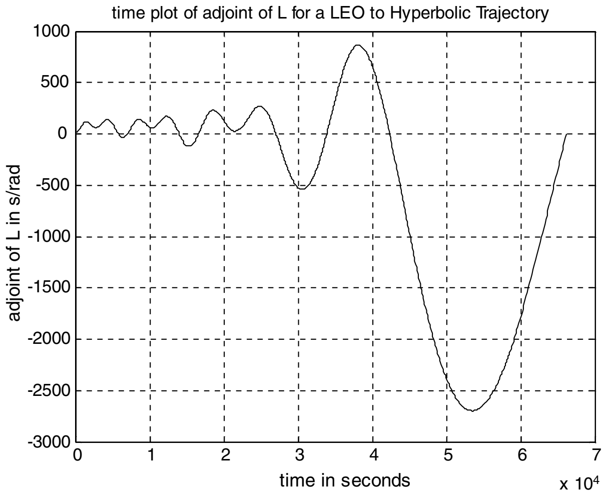

Fig. 1 λ_L **versus time.**

the iteration has a slightly higher eccentricity than that of the problem in [9]. The second iteration uses the solution obtained from the first iteration, and the eccentricity of the target orbit is again increased slightly from that of the previous iteration. The outlined procedure is repeated until the transfer problem is solved for $e = 1.1$. The increment of eccentricity is 0.1 for the first homotopy step and is reduced to 0.05 for the subsequent steps. It follows that a total of 21 homotopy steps are used to solve the problem. The outlined iterative procedure yields the following solution: $t_f = 6192.914215953497$ s, $\lambda_p(t_0) = 4.8495448249895343$ s/km, $\lambda_f(t_0) = -1426.3833992063583$ s, $\lambda_g(t_0) = 121.19890110385845$ s, $\lambda_h(t_0) = -54850.548574064494$ s, $\lambda_k(t_0) = 15887.730027143522$, and $L(t_0) = -3.2774971170996596$ rad.

The Hamiltonian for the system with the solution given here is very close to unity at the end of flight, $H(t_f) - 1 = 1.1354 \times 10^{-10}$, thus demonstrating that the control vector **P** satisfies the first-order optimality condition. It is important to note that the invariance of H with respect to time cannot be used as an indicator for optimality because the gravitational perturbations from the sun and the moon are explicit functions of time. This implies that the optimal trajectory's Hamiltonian is an explicit function of time; hence, $\mathrm{d}H/\mathrm{d}t$ is not zero [5]. Figure 1 is the time history of the adjoint of L. As shown in the figure, both the initial and final values of λ_L are 0; therefore, the free–free conditions for L are met. Plots for the time histories of the Hamiltonion, state variables, and other adjoint variables are given in [10].

Conclusions

To obtain the analytical forms of the MEE-based optimal control equations, the Jacobian matrix for the ECI state vector was derived, and its elements are presented here. An example problem involving the design of an escape trajectory under the effects of higher-order Earth gravity terms and sun–moon perturbations using the matrix is given. The equations for the elements of the Jacobian matrix will be especially applicable for the design of escape trajectories using any indirect method, and the example problem serves as a baseline case for further investigations such as studies on the use of more realistic thrust acceleration or additional gravitational perturbations.

Appendix: Elements of $\frac{\partial \mathbf{r}}{\partial \mathbf{Z}}$ and $\frac{\partial \mathbf{v}}{\partial \mathbf{Z}}$

The variables used in the equations that follow are the same as those used in [3].

$$q = 1 + f\cos L + g\sin L, \qquad r = \frac{p}{q}$$

$$s^2 = 1 + h^2 + k^2, \qquad \alpha^2 = h^2 - k^2$$

$$X = A_X B_X, \quad \text{where } A_X = \frac{r}{s^2}$$

$$B_X = \cos(L) + \alpha^2\cos(L) + 2hk\sin(L)$$

$$\frac{\partial X}{\partial p} = \frac{1}{qs^2}[\cos(L) + \alpha^2\cos(L) + 2hk\sin(L)]$$

$$\frac{\partial X}{\partial f} = -\frac{p}{(qs)^2}\cos(L)[\cos(L) + \alpha^2\cos(L) + 2hk\sin(L)]$$

$$\frac{\partial X}{\partial g} = -\frac{p}{(qs)^2}\sin(L)[\cos(L) + \alpha^2\cos(L) + 2hk\sin(L)]$$

$$\frac{\partial X}{\partial h} = \frac{\partial A_X}{\partial h}B_X + A_X\frac{\partial B_X}{\partial h} \qquad \frac{\partial X}{\partial k} = \frac{\partial A_X}{\partial k}B_X + A_X\frac{\partial B_X}{\partial k}$$

$$\frac{\partial X}{\partial L} = \frac{\partial A_X}{\partial L}B_X + A_X\frac{\partial B_X}{\partial L}, \qquad \frac{\partial A_X}{\partial h} = -\frac{2rh}{s^4}$$

$$\frac{\partial A_X}{\partial k} = -\frac{2rk}{s^4} \qquad \frac{\partial A_X}{\partial L} = \frac{p}{(qs)^2}[f\cdot\sin(L) - g\cdot\cos(L)]$$

$$\frac{\partial B_X}{\partial h} = 2\cdot[h\cos(L) + k\sin(L)]$$

$$\frac{\partial B_X}{\partial k} = 2\cdot[h\sin(L) - k\cos(L)]$$

$$\frac{\partial B_X}{\partial L} = [2hk\cos(L) - (1 + \alpha^2)\sin(L)]$$

$$Y = A_Y B_Y, \quad \text{where } A_Y = A_X = \frac{r}{s^2}$$

$$B_Y = (1 - \alpha^2)\sin(L) + 2hk\cos(L)$$

$$\frac{\partial Y}{\partial p} = \frac{1}{qs^2}[(1 - \alpha^2)\sin(L) + 2hk\cos(L)]$$

$$\frac{\partial Y}{\partial f} = -\frac{p}{(qs)^2}\cos(L)[(1 - \alpha^2)\sin(L) + 2hk\cos(L)]$$

$$\frac{\partial Y}{\partial g} = -\frac{p}{(qs)^2}\sin(L)[(1 - \alpha^2)\sin(L) + 2hk\cos(L)]$$

$$\frac{\partial Y}{\partial h} = \frac{\partial A_Y}{\partial h}B_Y + A_Y\frac{\partial B_Y}{\partial h} \qquad \frac{\partial Y}{\partial k} = \frac{\partial A_Y}{\partial k}B_Y + A_Y\frac{\partial B_Y}{\partial k}$$

$$\frac{\partial Y}{\partial L} = \frac{\partial A_Y}{\partial L}B_Y + A_Y\frac{\partial B_Y}{\partial L}, \qquad \frac{\partial B_Y}{\partial h} = 2\cdot[k\cos(L) - h\sin(L)]$$

$$\frac{\partial B_Y}{\partial k} = 2\cdot[k\sin(L) + h\cos(L)]$$

$$\frac{\partial B_Y}{\partial L} = [(1 - \alpha^2)\cos(L) - 2hk\sin(L)]$$

$$Z = 2A_Z B_Z, \quad \text{where, } A_Z = A_X = A_Y = \frac{r}{s^2}$$

$$B_Z = h\sin(L) - k\cos(L)$$

$$\frac{\partial Z}{\partial p} = \frac{2}{qs^2}[h\sin(L) - k\cos(L)]$$

$$\frac{\partial Z}{\partial f} = -\frac{2p}{(qs)^2}\cos(L)[h\sin(L) - k\cos(L)]$$

$$\frac{\partial Z}{\partial g} = -\frac{2p}{(qs)^2}\sin(L)[h\sin(L) - k\cos(L)]$$

$$\frac{\partial Z}{\partial h} = 2\cdot\left(\frac{\partial A_Z}{\partial h}B_Z + A_Z\frac{\partial B_Z}{\partial h}\right)$$

$$\frac{\partial Z}{\partial k} = 2\cdot\left(\frac{\partial A_Z}{\partial k}B_Z + A_Z\frac{\partial B_Z}{\partial k}\right)$$

$$\frac{\partial Z}{\partial L} = 2\cdot\left(\frac{\partial A_Z}{\partial L}B_Z + A_Z\frac{\partial B_Z}{\partial L}\right)$$

$$\frac{\partial B_Z}{\partial h} = \sin(L), \qquad \frac{\partial B_Z}{\partial k} = -\cos(L)$$

$$\frac{\partial B_Z}{\partial L} = h\cos(L) + k\sin(L)$$

$$V_X = C_X D_X, \quad \text{where } C_X = -\frac{1}{s^2}\sqrt{\frac{\mu}{p}}$$

$$D_X = \sin(L) + \alpha^2 \sin(L) - 2hk\cos(L) - 2fhk + \alpha^2 g + g$$

$$\frac{\partial V_X}{\partial p} = \frac{\partial C_X}{\partial p} D_X + C_X \frac{\partial D_X}{\partial p}, \qquad \frac{\partial V_X}{\partial f} = \frac{\partial C_X}{\partial f} D_X + C_X \frac{\partial D_X}{\partial f}$$

$$\frac{\partial V_X}{\partial g} = \frac{\partial C_X}{\partial g} D_X + C_X \frac{\partial D_X}{\partial g} \qquad \frac{\partial V_X}{\partial h} = \frac{\partial C_X}{\partial h} D_X + C_X \frac{\partial D_X}{\partial h}$$

$$\frac{\partial V_X}{\partial k} = \frac{\partial C_X}{\partial k} D_X + C_X \frac{\partial D_X}{\partial k}, \qquad \frac{\partial V_X}{\partial L} = \frac{\partial C_X}{\partial L} D_X + C_X \frac{\partial D_X}{\partial L}$$

$$\frac{\partial C_X}{\partial p} = \frac{1}{2ps^2}\sqrt{\frac{\mu}{p}}, \qquad \frac{\partial C_X}{\partial f} = \frac{\partial C_X}{\partial g} = \frac{\partial C_X}{\partial L} = 0$$

$$\frac{\partial C_X}{\partial h} = \frac{2h}{s^4}\sqrt{\frac{\mu}{p}}, \qquad \frac{\partial C_X}{\partial k} = \frac{2k}{s^4}\sqrt{\frac{\mu}{p}}, \qquad \frac{\partial D_X}{\partial p} = 0$$

$$\frac{\partial D_X}{\partial f} = -2hk, \qquad \frac{\partial D_X}{\partial g} = 1 + \alpha^2$$

$$\frac{\partial D_X}{\partial h} = 2h[\sin(L) + g] - 2k\cos(L) - 2fk$$

$$\frac{\partial D_X}{\partial k} = -2k[\sin(L) + g] - 2h\cos(L) - 2fh$$

$$\frac{\partial D_X}{\partial L} = (1 + \alpha^2)\cos(L) + 2hk\sin(L)$$

$$V_Y = C_Y D_Y, \quad \text{where } C_Y = C_X = -\frac{1}{s^2}\sqrt{\frac{\mu}{p}}$$

$$D_Y = (\alpha^2 - 1)\cos(L) + 2hk\sin(L) + (\alpha^2 - 1)f + 2ghk$$

$$\frac{\partial V_Y}{\partial p} = \frac{\partial C_Y}{\partial p} D_Y + C_Y \frac{\partial D_Y}{\partial p}, \qquad \frac{\partial V_Y}{\partial f} = \frac{\partial C_Y}{\partial f} D_Y + C_Y \frac{\partial D_Y}{\partial f}$$

$$\frac{\partial V_Y}{\partial g} = \frac{\partial C_Y}{\partial g} D_Y + C_Y \frac{\partial D_Y}{\partial g} \qquad \frac{\partial V_Y}{\partial h} = \frac{\partial C_Y}{\partial h} D_Y + C_Y \frac{\partial D_Y}{\partial h}$$

$$\frac{\partial V_Y}{\partial k} = \frac{\partial C_Y}{\partial k} D_Y + C_Y \frac{\partial D_Y}{\partial k}, \qquad \frac{\partial V_Y}{\partial L} = \frac{\partial C_Y}{\partial L} D_Y + C_Y \frac{\partial D_Y}{\partial L}$$

$$\frac{\partial D_Y}{\partial p} = 0, \qquad \frac{\partial D_Y}{\partial f} = \alpha^2 - 1, \qquad \frac{\partial D_Y}{\partial g} = 2hk$$

$$\frac{\partial D_Y}{\partial h} = 2h[\cos(L) + f] + 2k\sin(L) + 2gk$$

$$\frac{\partial D_Y}{\partial k} = -2k[\cos(L) + f] + 2h\sin(L) + 2gh$$

$$\frac{\partial D_Y}{\partial L} = (1 - \alpha^2)\sin(L) + 2hk\cos(L)$$

$$V_Z = -2C_Z D_Z, \quad \text{where } C_Z = C_Y = C_X = -\frac{1}{s^2}\sqrt{\frac{\mu}{p}}$$

$$D_Z = h[\cos(L) + f] + k[\sin(L) + g]$$

$$\frac{\partial V_Z}{\partial p} = -2\cdot\left(\frac{\partial C_Z}{\partial p} D_Z + C_Z \frac{\partial D_Z}{\partial p}\right)$$

$$\frac{\partial V_Z}{\partial f} = -2\cdot\left(\frac{\partial C_Z}{\partial f} D_Z + C_Z \frac{\partial D_Z}{\partial f}\right)$$

$$\frac{\partial V_Z}{\partial g} = -2\cdot\left(\frac{\partial C_Z}{\partial g} D_Z + C_Z \frac{\partial D_Z}{\partial g}\right)$$

$$\frac{\partial V_Z}{\partial h} = -2\cdot\left(\frac{\partial C_Z}{\partial h} D_Z + C_Z \frac{\partial D_Z}{\partial h}\right)$$

$$\frac{\partial V_Z}{\partial k} = -2\cdot\left(\frac{\partial C_Z}{\partial k} D_Z + C_Z \frac{\partial D_Z}{\partial k}\right)$$

$$\frac{\partial V_Z}{\partial L} = -2\cdot\left(\frac{\partial C_Z}{\partial L} D_Z + C_Z \frac{\partial D_Z}{\partial L}\right)$$

$$\frac{\partial D_Z}{\partial p} = 0, \qquad \frac{\partial D_Z}{\partial f} = h, \qquad \frac{\partial D_Z}{\partial g} = k$$

$$\frac{\partial D_Z}{\partial h} = \cos(L) + f, \qquad \frac{\partial D_Z}{\partial k} = \sin(L) + g$$

$$\frac{\partial D_Z}{\partial L} = k\cos(L) - h\sin(L)$$

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